Dynamics of a Cantilever Beam Attached to a Moving Base

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The behavior of a cantilever beam built into a rigid body that is performing a specified motion of rotation and translation is studied with two objectives in mind. First, because the subject is of interest in connection with spacecraft antennae, helicopter rotor blades, robot arms, and other systems that perform complex motions, we create an algorithm that can be used to predict the behavior of the beam when the base undergoes general three-dimensional motions. Effects such as centrifugal stiffening and vibrations induced by Coriolis forces are accommodated automatically, rather than with the aid of ad hoc provisions. The second objective is to draw attention to fundamental flaws in certain multibody computer programs currently under development or already in use. To this end, we construct a second simulation algorithm, one that embodies the procedure apparently employed in the programs in question, and then compare simulation results produced by computer programs based on the two algorithms. Conflicts between the two approaches that thus come to light are discussed in detail.

I. Introduction

STUDY of the behavior of flexible bodies attached to moving supports has been vigorously pursued for over fifty years in connection with a number of diverse disciplines, such as machine design, robotics, aircraft dynamics, and spacecraft dynamics. In particular, beams attached to moving bases have received attention in hundreds of technical papers dealing variously with elastic linkages, rotating machinery, robotic manipulator arms, aircraft propellors, helicopter rotor blades, and flexible satellites. Indeed, the existing literature is so voluminous as to preclude comprehensive review within the confines of a technical paper. (A 1974 review, by Modi,[1] of the literature on rigid bodies with flexible appendages contained more than two hundred references.) Accordingly, we shall mention only representative publications that permit us to direct attention to issues of primary concern.

In the area of machine design, the published analyses[2-5] deal mainly with planar bending of rotating links or with pure torsion of shafted systems. For the most part, the goal of these analyses is to determine forces or stresses of interest in connection with machine component sizing. In robotics,[6-9] displacements, such as those giving rise to bending of a manipulator arm, are at the center of attention, and various beam effects are frequently neglected in an attempt to increase on-line computational speed. Analyses performed in connection with aircraft dynamics[10-14] address questions concerning tapered, twisted, and rotating beams; and in the field of spacecraft dynamics,[15-21] the effect of vehicle elasticity on attitude motions is of particular interest.

What is missing from this vast literature is a comprehensive theory for dealing with small vibrations of a general beam attached to a base that is undergoing an arbitrary, prescribed motion (translation and rotation). By a general beam we mean one that has an arbitrary cross section and material properties that vary from point to point along the beam; and a compre-

hensive theory is one that accommodates all of the following: extension, bending in two principal planes, torsion (uniform or nonuniform), shear displacements, and warping.

In the existing literature, one often finds that a particular effect is introduced in an ad hoc fashion. For example, there are many analyses that make special provisions for centrifugal stiffening, Coriolis forces, etc. Most of these analyses are based on one of two methodologies. The first[22] involves starting with equations applicable to a beam with an inertially fixed support, and then adding terms intended to reflect an effect of interest, such as the Coriolis acceleration associated with a rotational motion of the base that supports the beam. In the second approach,[21,23] which is based on a Rayleigh-Ritz technique, one works with trial functions or assumed modes, as these functions are often called, some of which are directly associated with the gyroscopic effects or centrifugal effects present in a beam performing a simple spinning motion at a constant spin rate. These two approaches share a basic shortcoming: they do not account, in a general way, for arbitrary base motion.

While most previous analyses deal either with spinning beams or beams attached to a base undergoing rectilinear translation, some attempts to construct more comprehensive theories have been undertaken. For example, Benedetti,[24] formulating equations of motion for a uniform Euler-Bernoulli beam with a symmetric cross section, let the base supporting the beam move with arbitrary angular and translational accelerations; however, terms associated with stiffening due to rotation were neglected because the analysis was intended for the study of the behavior of flexible projectiles, in which connection it is appropriate to drop these terms since the dominant angular velocity component is parallel to the axis of the projectile. The fact that this work is restricted to beams with symmetric cross sections precludes its use for the purpose of analyzing motions of open section beams, such as STEM (self-storing tubular extensible module) booms, which are commonly used on satellites. Laskin et al.[25] also attempted to develop a general theory for a free-free, symmetric cross section Rayleigh beam, but stopped short of the mark and developed final equations only for the Bernoulli-Euler beam with a quasi-steady-state stretch.

Modi ended his literature review with the suggestion that in future work "more attention should be directed towards evolution of general analyses rather than exploration of specific

configurations." It is the principal objective of this paper to present such an analysis. In addition, we wish to draw attention to the fact that rather subtle errors can lead to theories with serious defects. Specifically, there exist multibody computer programs, such as DISCOS,[26] NBOD2,[27] ALLFLEX,[28,29] and TREETOPS,[30] for which a claim is made in the associated documentation that these programs can produce accurate simulation results of "large" motions of systems containing flexible bodies that are themselves concurrently undergoing "small" deformations. In fact, unless ad hoc corrective measures are taken, these programs can produce totally incorrect results in certain situations.

The remainder of this paper is organized as follows. The system to be studied is described in the next section. In Sec. III, the new theory is set forth. An associated algorithm that can be implemented on a digital computer is described in Sec. IV. An illustrative example involving a robot manipulator arm with general base motion is presented in Sec. V, and the present approach is compared with the "conventional" one in Sec. VI. Finally, in Sec. VII, conclusions are drawn, and plans to incorporate the new theory into a multibody computer program are discussed.

II. System Description

The system to be analyzed, shown in Fig. 1, consists of a cantilever beam B built into a rigid body A whose motion in a Newtonian reference frame N is prescribed as a function of time. The beam is characterized by a natural length L, material properties $E_o(x)$, $G(x)$, $\rho(x)$, and cross-sectional properties $A_o(x)$, $I_2(x)$, $I_3(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\kappa(x)$, $\Gamma(x)$, $e_2(x)$, and $e_3(x)$, defined as follows. Let x be the distance from a point O, located at the root of B, to the plane of a generic cross section of B, when B is undeformed. Then $E_o(x)$, $G(x)$, and $\rho(x)$ are the modulus of elasticity, the shear modulus, and the mass per unit length of the beam, respectively, as functions of x. The area of the cross section located at a distance x from point O is denoted as $A_o(x)$, and the Saint Venant torsion factor and the warping factor are represented by the symbols $\kappa(x)$ and $\Gamma(x)$, respectively. In order to define the symbols $I_2(x)$, $I_3(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $e_2(x)$, and $e_3(x)$, introduce a dextral set of mutually perpendicular unit vectors $\underline{b}_1, \underline{b}_2, \underline{b}_3$, fixed in the plane of the cross section located at a distance x from point O, and oriented such that $\underline{b}_1$ is parallel to the centroidal axis of B, while $\underline{b}_2$ and $\underline{b}_3$ are parallel to central principal axes of the cross section. Additionally, introduce a similar set of unit vectors $\underline{a}_1, \underline{a}_2, \underline{a}_3$, fixed in A and parallel, respectively, to $\underline{b}_1, \underline{b}_2, \underline{b}_3$ when B is undeformed. The symbols $I_2(x)$ and $I_3(x)$ denote the central principal second moments of area of the generic cross section for unit vectors $\underline{b}_2$ and $\underline{b}_3$, respectively, and $\alpha_2(x)$ and $\alpha_3(x)$ are the shear area ratios (also called shear correction factors, see Ref. 31, p. 351) of the

section for the $\underline{b}_2$ and $\underline{b}_3$ directions. The parameters $e_2(x)$ and $e_3(x)$ are measure numbers of the $\underline{b}_2$ and $\underline{b}_3$ components of the eccentricity vector, which extends from the elastic center of the generic cross section to the centroid. The elastic center, the flexural center, and the center of twist all are assumed to coincide in the present analysis.[32]

III. Equations of Motion

The goal of the following analysis is to produce equations governing the extension, transverse bending, shear, and torsion of the beam introduced in the previous section. The reader who is interested primarily in the resulting computational algorithm rather than details of the equation formulation should proceed directly to the following section. In the sequel, an underlined symbol denotes a vector, a doubly underlined symbol signifies a dyadic, and a symbol in boldface type represents a matrix. Numbers listed over equal signs refer to previous equations, and a triangle over an equal sign indicates a definition.

The motion of A in N is characterized by the six quantities v_1, v_2, v_3, ω_1, ω_2, and ω_3 defined as

$$v_i \overset{\triangle}{=} {}^N\!\underline{v}^O \cdot \underline{a}_i \qquad (i=1,2,3) \tag{1}$$

$$\omega_i \overset{\triangle}{=} {}^N\!\underline{\omega}^A \cdot \underline{a}_i \qquad (i=1,2,3) \tag{2}$$

where ${}^N\!\underline{v}^O$ is the inertial velocity of the point O (see Fig. 2) on the elastic axis§ of B at the root and ${}^N\!\underline{\omega}^A$ is the angular velocity of A in N. Equations (1) and (2) imply that

$$^N\!\underline{v}^O = v_1\underline{a}_1 + v_2\underline{a}_2 + v_3\underline{a}_3 \tag{3}$$

and

$$^N\!\underline{\omega}^A = \omega_1\underline{a}_1 + \omega_2\underline{a}_2 + \omega_3\underline{a}_3 \tag{4}$$

To describe the motion of B in N, introduce a rigid differential element dB of B, with centroid C and elastic center E located at a distance x from point O when B is undeformed. The differential element dB can be brought into a general orientation in A by aligning $\underline{b}_1, \underline{b}_2, \underline{b}_3$ with $\underline{a}_1, \underline{a}_2, \underline{a}_3$, respectively, and subjecting dB to successive dextral rotations in A of amounts θ_1, θ_2, and θ_3 about lines parallel to $\underline{a}$, $\underline{a}_2$, and $\underline{a}_3$, respectively. Similarly, point E of dB can be brought into a general position in A by subjecting E to successive translations $u_1\underline{a}_1$, $u_2\underline{a}_2$, and $u_3\underline{a}_3$. If the vectors $\underline{r}$ and $\underline{u}$ are defined as

$$\underline{r} \overset{\triangle}{=} x\underline{a}_1 \tag{5}$$

$$\underline{u} \overset{\triangle}{=} u_1\underline{a}_1 + u_2\underline{a}_2 + u_3\underline{a}_3 \tag{6}$$

then the position vector $\underline{p}^{OE}$ from O to E can be expressed as

$$\underline{p}^{OE} = \underline{r} + \underline{u} \tag{7}$$

The position vector $\underline{p}^{EC}$ from the elastic center E to the centroid C is the eccentricity vector; that is,

$$\underline{p}^{EC} = \underline{e} = e_2\underline{b}_2 + e_3\underline{b}_3 \tag{8}$$

Therefore, the position vector $\underline{p}^{OC}$ from O to C is given by

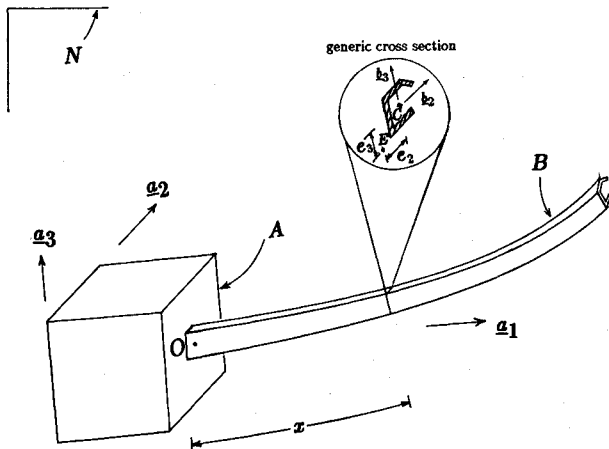$$\underline{p}^{OC} = \underline{p}^{OE} + \underline{p}^{EC} \tag{9}$$

Fig. 1   Beam attached to moving rigid base.

§The elastic axis is the line along which transverse loads must be applied in order to produce bending unaccompanied by torsion of the beam at any section.[32] This axis passes through the elastic center E of every section.
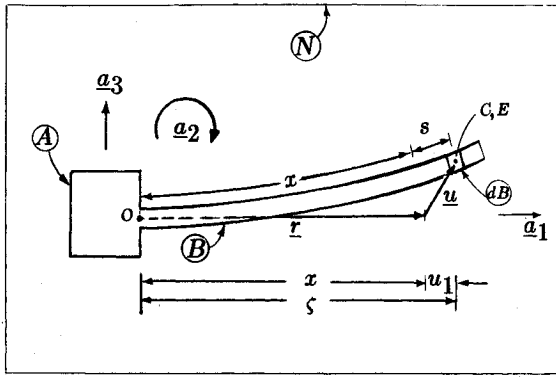
Fig. 2   Deformed beam with differential element.

$$p^{OC} \overset{(5-8)}{=} (x + u_1)\underline{a}_1 + u_2\underline{a}_2 + u_3\underline{a}_3 + e_2\underline{b}_2 + e_3\underline{b}_3 \quad (10)$$

The orientation of dB in A can be described by reference to a direction cosine matrix $^AC^{dB}$ such that the element C_{ij} in the ith row and jth column is defined as

$$C_{ij} \triangleq \underline{a}_i \cdot \underline{b}_j \qquad (i, j = 1, 2, 3) \quad (11)$$

The matrix $^AC^{dB}$, referred to as a space-three, 1-2-3 rotation matrix, is given by[33]

$$^AC^{dB} = \begin{bmatrix} c_2c_3 & s_1s_2c_3 - s_3c_1 & c_1s_2c_3 + s_3s_1 \\ c_2s_3 & s_1s_2s_3 + c_3c_1 & c_1s_2s_3 - c_3s_1 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix} \quad (12)$$

with c_i and s_i defined as

$$s_i \triangleq \sin\theta_i, \qquad c_i \triangleq \cos\theta_i \qquad (i = 1, 2, 3) \quad (13)$$

The velocity of C in N is found by using the relationship

$$^N\underline{v}^C = {}^N\underline{v}^O + {}^N\underline{\omega}^A \times p^{OC} + {}^A\underline{v}^C \quad (14)$$

where $^A\underline{v}^C$, the velocity of C in A, can be formed by differentiating the position vector p^{OC} with respect to time in reference frame A. Substituting the resulting expression for $^A\underline{v}^C$, along with the expressions for $^N\underline{v}^O$, $^N\underline{\omega}^A$, and p^{OC} from Eqs. (3), (4), and (10), respectively, into Eq. (14) yields

$$
\begin{aligned}
^N\underline{v}^C = \{ &v_1 + \omega_2[u_3 + e_2s_1c_2 + e_3c_1c_2] \\
&- \omega_3[u_2 + e_2(s_1s_2s_3 + c_3c_1) + e_3(c_1s_2s_3 - c_3s_1)] + \dot{u}_1 \\
&+ e_2[(c_1s_2c_3 + s_3s_1)\dot{\theta}_1 + s_1c_2c_3\dot{\theta}_2 - (s_1s_2s_3 + c_3c_1)\dot{\theta}_3] \\
&+ e_3[(-s_1s_2c_3 + s_3c_1)\dot{\theta}_1 + c_1c_2c_3\dot{\theta}_2 \\
&+ (-c_1s_2s_3 + c_3s_1)\dot{\theta}_3] \} \underline{a}_1 \\
+ \{ &v_2 + \omega_3[x + u_1 + e_2(s_1s_2c_3 - s_3c_1) \\
&+ e_3(c_1s_2c_3 + s_3s_1)] - \omega_1[u_3 + e_2s_1c_2 + e_3c_1c_2] + \dot{u}_2 \\
&+ e_2[(c_1s_2s_3 - c_3s_1)\dot{\theta}_1 + s_1c_2s_3\dot{\theta}_2 + (s_1s_2c_3 - s_3c_1)\dot{\theta}_3] \\
&+ e_3[-(s_1s_2s_3 + c_3c_1)\dot{\theta}_1 + c_1c_2s_3\dot{\theta}_2 + (c_1s_2c_3 + s_3s_1)\dot{\theta}_3]\} \underline{a}_2 \\
+ \{ &v_3 + \omega_1[u_2 + e_2(s_1s_2s_3 + c_3c_1) + e_3(c_1s_2s_3 - c_3s_1)] \\
&- \omega_2[x + u_1 + e_2(s_1s_2c_3 - s_3c_1) + e_3(c_1s_2c_3 + s_3s_1)] \\
&+ \dot{u}_3 + e_2[c_1c_2\dot{\theta}_1 - s_1s_2\dot{\theta}_2] - e_3[s_1c_2\dot{\theta}_1 + c_1s_2\dot{\theta}_2]\} \underline{a}_3
\end{aligned}
$$
$$(15)$$

The previous definition of θ_1, θ_2, and θ_3 leads to the following expression for the angular velocity of dB in A:

$$
\begin{aligned}
^A\underline{\omega}^{dB} = &\left(\dot{\theta}_1 - \dot{\theta}_3s_2\right)\underline{b}_1 + \left(\dot{\theta}_2c_1 + \dot{\theta}_3s_1c_2\right)\underline{b}_2 \\
&+ \left(-\dot{\theta}_2s_1 + \dot{\theta}_3c_1c_2\right)\underline{b}_3
\end{aligned}
\quad (16)
$$

The angular velocity of dB in N, found by using the angular velocity addition theorem,[34] is thus given by

$$
\begin{aligned}
^N\underline{\omega}^{dB} = &{}^N\underline{\omega}^A + {}^A\underline{\omega}^{dB} \\
\overset{(4,12,16)}{=} &\left[\omega_1c_2c_3 + \omega_2c_2s_3 - \omega_3s_2 + \dot{\theta}_1 - \dot{\theta}_3s_2\right]\underline{b}_1 \\
&+ \left[\omega_1(s_1s_2c_3 - s_3c_1) + \omega_2(s_1s_2s_3 + c_3c_1)\right. \\
&\left. + \omega_3s_1c_2 + \dot{\theta}_2c_1 + \dot{\theta}_3s_1c_2\right]\underline{b}_2 \\
&+ \left[\omega_1(c_1s_2c_3 + s_3s_1) + \omega_2(c_1s_2s_3 - c_3s_1)\right. \\
&\left. + \omega_3c_1c_2 - \dot{\theta}_2s_1 + \dot{\theta}_3c_1c_2\right]\underline{b}_3
\end{aligned}
\quad (17)
$$

The quantities $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$, $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ appearing in Eqs. (15) and (17) ultimately must be expressed in terms of system generalized coordinates. With this in mind, we introduce ζ as

$$\zeta \triangleq p^{OE} \cdot \underline{a}_1 \quad (18)$$

and let $s(x, t)$ denote the stretch in the beam along the elastic axis; then the distance (see Fig. 2) along the deformed elastic axis from O to point E, which point is at a distance x from O when B is undeformed, is

$$x + s(x, t) = \int_0^\zeta \left\{1 + \left[\frac{\partial u_2(\sigma, t)}{\partial \sigma}\right]^2 + \left[\frac{\partial u_3(\sigma, t)}{\partial \sigma}\right]^2\right\}^{\frac{1}{2}} d\sigma \quad (19)$$

Now it is convenient to define $J(\sigma, t)$ as

$$J(\sigma, t) \triangleq 1 + \left[\frac{\partial u_2(\sigma, t)}{\partial \sigma}\right]^2 + \left[\frac{\partial u_3(\sigma, t)}{\partial \sigma}\right]^2 \quad (20)$$

whereupon differentiation of Eq. (19) with respect to time leads to

$$\dot{s}(x, t) = \frac{1}{2}\int_0^\zeta [J(\sigma, t)]^{-\frac{1}{2}}\frac{\partial J(\sigma, t)}{\partial t} d\sigma + \dot{\zeta}[J(\zeta, t)]^{\frac{1}{2}} \quad (21)$$

and solution of Eq. (21) for $\dot{\zeta}$ produces

$$\dot{\zeta} = \frac{-\frac{1}{2}\int_0^\zeta [J(\sigma, t)]^{-\frac{1}{2}}[\partial J(\sigma, t)/\partial t] d\sigma + \dot{s}(x, t)}{[J(\zeta, t)]^{\frac{1}{2}}} \quad (22)$$

From Eqs. (5-7) and (18), it is apparent that

$$\zeta = x + u_1 \quad (23)$$

Hence,

$$\dot{u}_1 \overset{(23)}{=} \dot{\zeta} \overset{(22)}{=} \frac{-\frac{1}{2}\int_0^\zeta [J(\sigma, t)]^{-\frac{1}{2}}[\partial J(\sigma, t)/\partial t] d\sigma + \dot{s}(x, t)}{[J(\zeta, t)]^{\frac{1}{2}}}$$
$$(24)$$

If the quantities $s(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $\theta_1(x,t)$, $\theta_2(x,t)$, and $\theta_3(x,t)$ are expressed¶ as

$$s(x,t) = \sum_{i=1}^{\nu} \phi_{1i}(x) q_i(t) \tag{25}$$

$$u_j(x,t) = \sum_{i=1}^{\nu} \phi_{ji}(x) q_i(t) \qquad (j = 2,3) \tag{26}$$

$$\theta_j(x,t) = \sum_{i=1}^{\nu} \phi_{j+3,i}(x) q_i(t) \qquad (j = 1,2,3) \tag{27}$$

where ϕ_{ij} ($i = 1,\ldots,6$; $j = 1,\ldots,\nu$) are as yet totally unrestricted spatial functions, q_i ($i = 1,\ldots,\nu$) are generalized coordinates, and ν is an arbitrary integer, then the term $\partial J(\sigma,t)/\partial t$ which appears in Eq. (24) can be written as

$$\frac{\partial J(\sigma,t)}{\partial t} \overset{(20,26)}{=} 2 \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \left[\phi'_{2i}(\sigma) \phi'_{2j}(\sigma) \right.$$
$$\left. + \phi'_{3i}(\sigma) \phi'_{3j}(\sigma) \right] \dot{q}_i(t) q_j(t) \tag{28}$$

where a prime denotes differentiation with respect to the dummy variable σ. Substituting this expression along with the one for $s(x,t)$ from Eq. (25) into Eq. (24) yields

$$\dot{u}_1 = [J(\zeta,t)]^{-\frac{1}{2}} \sum_{i=1}^{\nu} \left\{ \phi_{1i}(x) - \sum_{j=1}^{\nu} \int_0^{\zeta} [J(\sigma,t)]^{-\frac{1}{2}} \right.$$
$$\left. \times \left[\phi'_{2i}(\sigma) \phi'_{2j}(\sigma) + \phi'_{3i}(\sigma) \phi'_{3j}(\sigma) \right] d\sigma \, q_j(t) \right\} \dot{q}_i(t) \tag{29}$$

Expressions for the remaining time derivatives appearing in Eqs. (15) and (17) follow directly from Eqs. (26–27). Specifically,

$$\dot{u}_j(x,t) = \sum_{i=1}^{\nu} \phi_{ji}(x) \dot{q}_i(t) \qquad (j = 2,3) \tag{30}$$

$$\dot{\theta}_j(x,t) = \sum_{i=1}^{\nu} \phi_{j+3,i}(x) \dot{q}_i(t) \qquad (j = 1,2,3) \tag{31}$$

and substitution of the expressions for $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$, $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ from Eqs. (29–31) into Eq. (15) results in

$$^N\underline{v}^C = \Big(v_1 + \omega_2 [u_3 + e_2 s_1 c_2 + e_3 c_1 c_2]$$
$$- \omega_3 [u_2 + e_2(s_1 s_2 s_3 + c_3 c_1) + e_3(c_1 s_2 s_3 - c_3 s_1)]$$
$$+ [J(\zeta,t)]^{-\frac{1}{2}} \sum_{i=1}^{\nu} \Big\{ \phi_{1i}(x) - \sum_{j=1}^{\nu} \int_0^{\zeta} [J(\sigma,t)]^{-\frac{1}{2}}$$
$$\times \left[\phi'_{2i}(\sigma) \phi'_{2j}(\sigma) + \phi'_{3i}(\sigma) \phi'_{3j}(\sigma) \right] d\sigma \, q_j \Big\} \dot{q}_i$$
$$+ e_2 \sum_{i=1}^{\nu} \{ [c_1 s_2 c_3 + s_3 s_1] \phi_{4i}(x) + [s_1 c_2 c_3] \phi_{5i}(x)$$
$$- [s_1 s_2 s_3 + c_3 c_1] \phi_{6i}(x) \} \dot{q}_i$$

$$+ e_3 \sum_{i=1}^{\nu} \{ [-s_1 s_2 c_3 + s_3 c_1] \phi_{4i}(x) + [c_1 c_2 c_3] \phi_{5i}(x)$$
$$+ [-c_1 s_2 s_3 + c_3 s_1] \phi_{6i}(x) \} \dot{q}_i \Big) \underline{a}_1 + \Big(v_2 + \omega_3 [x + u_1$$
$$+ e_2(s_1 s_2 c_3 - s_3 c_1) + e_3(c_1 s_2 c_3 + s_3 s_1)] \Big)$$
$$- \omega_1 [u_3 + e_2 s_1 c_2 + e_3 c_1 c_2] + \sum_{i=1}^{\nu} \phi_{2i}(x) \dot{q}_i$$
$$+ e_2 \sum_{i=1}^{\nu} \{ [c_1 s_2 s_3 - c_3 s_1] \phi_{4i}(x) + [s_1 c_2 s_3] \phi_{5i}(x)$$
$$+ [s_1 s_2 c_3 - s_3 c_1] \phi_{6i}(x) \} \dot{q}_i$$
$$+ \Big(e_3 \sum_{i=1}^{\nu} \{ -[s_1 s_2 s_3 + c_3 c_1] \phi_{4i}(x) + [c_1 c_2 s_3] \phi_{5i}(x)$$
$$+ [c_1 s_2 c_3 + s_3 s_1] \phi_{6i}(x) \} \dot{q}_i \Big) \underline{a}_2$$
$$+ \Big(v_3 + \omega_1 [u_2 + e_2(s_1 s_2 s_3 + c_3 c_1) + e_3(c_1 s_2 s_3 - c_3 s_1)]$$
$$- \omega_2 [x + u_1 + e_2(s_1 s_2 c_3 - s_3 c_1) + e_3(c_1 s_2 c_3 + s_3 s_1)]$$
$$+ \sum_{i=1}^{\nu} \phi_{3i}(x) \dot{q}_i + e_2 \sum_{i=1}^{\nu} \{ [c_1 c_2] \phi_{4i}(x)$$
$$- [s_1 s_2] \phi_{5i}(x) \} \dot{q}_i$$
$$- e_3 \sum_{i=1}^{\nu} \{ [s_1 c_2] \phi_{4i}(x) + [c_1 s_2] \phi_{5i}(x) \} \dot{q}_i \Big) \underline{a}_3 \tag{32}$$

while substituting the expressions for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ from Eq. (31) into Eq. (17) yields

$$^N\underline{\omega}^{dB} = \Big\{ \omega_1 c_2 c_3 + \omega_2 c_2 s_3 - \omega_3 s_2$$
$$+ \sum_{i=1}^{\nu} [\phi_{4i}(x) - s_2 \phi_{6i}(x)] \dot{q}_i \Big\} \underline{b}_1$$
$$+ \Big\{ \omega_1(s_1 s_2 c_3 - s_3 c_1) + \omega_2(s_1 s_2 s_3 + c_3 c_1) + \omega_3 s_1 c_2$$
$$+ \sum_{i=1}^{\nu} [\phi_{5i}(x) c_1 + \phi_{6i}(x) s_1 c_2] \dot{q}_i \Big\} \underline{b}_2$$
$$+ \Big\{ \omega_1(c_1 s_2 c_3 + s_3 s_1) + \omega_2(c_1 s_2 s_3 - c_3 s_1) + \omega_3 c_1 c_2$$
$$+ \sum_{i=1}^{\nu} [-\phi_{5i}(x) s_1 + \phi_{6i}(x) c_1 c_2] \dot{q}_i \Big\} \underline{b}_3 \tag{33}$$

Terms referred to as partial velocities (Ref. 34, p. 45) of C in N, and partial angular velocities of dB in N will be denoted by the symbols $^N\underline{v}_i^C$ and $^N\underline{\omega}_i^{dB}$ ($i = 1,\ldots,\nu$), respectively, and will play a major role in the subsequent formulation of equations of motion. The partial velocity $^N\underline{v}_i^C$ is defined as the coefficient of $\dot{q}_i$ ($i = 1,\ldots,\nu$) in the velocity expression shown in Eq. (32), and the partial angular velocity $^N\underline{\omega}_i^{dB}$ is similarly defined as the coefficient of $\dot{q}_i$ ($i = 1,\ldots,\nu$) in the angular velocity expression given in Eq. (33). Consequently, these partial velocities and partial angular velocities

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¶The assumed mode method is employed here with a concomitant assumption that the displacement functions can be separated into products of spatial and temporal functions.

can be written

$$\overset{N}{\underset{i}{\upsilon}}{}_{i}^{C} \overset{(32)}{=} [J(\zeta,t)]^{-\frac{1}{2}}\Bigg\{ \phi_{1i}(x) - \sum_{j=1}^{\nu} \int_{0}^{\zeta}[J(\sigma,t)]^{-\frac{1}{2}}$$

$$\times \Big[\phi'_{2i}(\sigma)\phi'_{2j}(\sigma) + \phi'_{3i}(\sigma)\phi'_{3j}(\sigma)\Big] d\sigma \, q_j \Bigg\}$$

$$+ e_2\{[c_1 s_2 c_3 + s_3 s_1]\phi_{4i}(x) + [s_1 c_2 c_3]\phi_{5i}(x)$$

$$- [s_1 s_2 s_3 + c_3 c_1]\phi_{6i}(x)\} + e_3\{[-s_1 s_2 c_3 + s_3 c_1]\phi_{4i}(x)$$

$$+ [c_1 c_2 c_3]\phi_{5i}(x) + [-c_1 s_2 s_3 + c_3 s_1]\phi_{6i}(x)\}\Big] \underline{a}_1$$

$$+ [\phi_{2i}(x) + e_2\{[c_1 s_2 s_3 - c_3 s_1]\phi_{4i}(x)]$$

$$+ [s_1 c_2 s_3]\phi_{5i}(x) + [s_1 s_2 c_3 - s_3 c_1]\phi_{6i}(x)\}$$

$$+ e_3\{-[s_1 s_2 s_3 + c_3 c_1]\phi_{4i}(x)$$

$$+ [c_1 c_2 s_3]\phi_{5i}(x) + [c_1 s_2 c_3 + s_3 s_1]\phi_{6i}(x)\}\underline{a}_2$$

$$+ (\phi_{3i}(x) + e_2\{[c_1 c_2]\phi_{4i}(x) - [s_1 s_2]\phi_{5i}(x)\}$$

$$- e_3\{[s_1 c_2]\phi_{4i}(x) + [c_1 s_2]\phi_{5i}(x)\})\underline{a}_3$$

$$(i = 1,\ldots,\nu) \quad (34)$$

and

$$\overset{N}{\underset{i}{\omega}}{}^{dB} \overset{(33)}{=} [\phi_{4i}(x) - s_2\phi_{6i}(x)]\underline{b}_1$$

$$+ [\phi_{5i}(x)c_1 + \phi_{6i}(x)s_1 c_2]\underline{b}_2$$

$$+ [-\phi_{5i}(x)s_1 + \phi_{6i}(x)c_1 c_2]\underline{b}_3 \quad (i = 1,\ldots,\nu) \quad (35)$$

Since the goal of this analysis is to produce equations governing motions of B in N during which $s(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $\theta_1(x,t)$, $\theta_2(x,t)$, and $\theta_3(x,t)$ remain small, it is appropriate at this point in the analysis to linearize *all* kinematic quantities in the variables $q_1,\ldots,q_\nu,\dot{q}_1,\ldots,\dot{q}_\nu$. When linearizations are performed subsequent to the formation of expressions for partial velocities and partial angular velocities, as is being done here, the resulting dynamical equations are identical to those that would be obtained if linearizations were performed after completely nonlinear equations have been formulated (Ref. 33, p. 277). The great advantage of linearizing at the present stage of the analysis is that this relieves one of the considerable burden of developing complicated nonlinear expressions for translational and angular accelerations. It is tempting to linearize still earlier, that is, prior to the formation of expressions for partial velocities and partial angular velocities; but such linearization would be premature and would lead to dynamical equations lacking certain "dynamic stiffness" terms. In the following analysis, we indicate that a quantity has been linearized in the generalized coordinates and their time derivatives by placing a tilde over the associated symbol.

When linearizing velocity and partial velocity expressions, it is beneficial to note that

$$J(x,t) \overset{(20,26)}{=} 1 + \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \Big[\phi'_{2i}(x)\phi'_{2j}(x) + \phi'_{3i}(x)\phi'_{3j}(x)\Big] q_i q_j \quad (36)$$

from which it follows that

$$\tilde{J}(x,t) = 1, \qquad \tilde{J}(\zeta,t) = 1 \quad (37)$$

The trigonometric quantities s_i and c_i $(i = 1,2,3)$ defined in Eq. (13) can be linearized by expanding in Maclaurin's series and dropping terms of degree 2 and higher in the variables $q_1,\ldots,q_\nu$.

An expression for $\tilde{u}_1$ is formed as follows. From Eqs. (19) and (20), write

$$x + s(x,t) = \int_{0}^{\zeta}[J(\sigma,t)]^{\frac{1}{2}} d\sigma \quad (38)$$

Linearize both sides of Eq. (38), which yields

$$x + \tilde{s}(x,t) \overset{(37)}{=} \tilde{\zeta} \overset{(23)}{=} x + \tilde{u}_1 \quad (39)$$

so that

$$\tilde{u}_1 \overset{(39)}{=} \tilde{s}(x,t) \overset{(25)}{=} \sum_{i=1}^{\nu} \phi_{1i}(x) q_i(t) \quad (40)$$

Define two new symbols, β_{ij} and γ_{ij}, as

$$\beta_{ij} \triangleq \int_{0}^{\zeta}\phi'_{2i}(\sigma)\phi'_{2j}(\sigma)\,d\sigma$$

$$\gamma_{ij} \triangleq \int_{0}^{\zeta}\phi'_{3i}(\sigma)\phi'_{3j}(\sigma)\,d\sigma \quad (i,j = 1,\ldots,\nu) \quad (41)$$

and express the linearized partial velocity of C in N as

$$\overset{N}{\underset{i}{\tilde{\upsilon}}}{}^{C} \overset{(34,36-41)}{=} \Bigg(\phi_{1i}(x) - \sum_{j=1}^{\nu}(\beta_{ij} + \gamma_{ij})q_j$$

$$+ e_2\Big\{\sum_{j=1}^{\nu} \big[\phi_{4i}(x) + \phi_{5j}(x) + \phi_{5i}(x)\phi_{4j}(x)\big]q_j - \phi_{6i}(x)\Big\}$$

$$+ e_3\Big\{\sum_{j=1}^{\nu}\big[\phi_{4i}(x)\phi_{6j}(x) + \phi_{6i}(x)\phi_{4j}(x)\big]q_j$$

$$+ \phi_{5i}(x)\Big\}\Bigg)\underline{a}_1 + \Bigg(\phi_{2i}(x) - e_2\sum_{j=1}^{\nu}\{\phi_{4i}(x)\phi_{4j}(x)$$

$$+ \phi_{6i}(x)\phi_{6j}(x)\}q_j\Bigg) + \Bigg(e_3\Big\{\sum_{j=1}^{\nu}\big[\phi_{5i}(x)\phi_{6j}(x)$$

$$+ \phi_{6i}(x)\phi_{5j}(x)\big]q_j - \phi_{4i}(x)\Big\}\Bigg)\underline{a}_2$$

$$+ \Bigg(\phi_{3i}(x) + e_2\phi_{4i}(x) - e_3\sum_{j=1}^{\nu}\{\phi_{4i}(x)\phi_{4j}(x)$$

$$+ \phi_{5i}(x)\phi_{5j}(x)\}q_j\Bigg)\underline{a}_3 \quad (i = 1,\ldots,\nu) \quad (42)$$

The linearized partial angular velocity of dB in N is

$$\overset{N}{\underset{i}{\tilde{\omega}}}{}^{dB} \overset{(35-41)}{=} \Bigg[\phi_{4i}(x) - \sum_{j=1}^{\nu}\phi_{5j}(x)\phi_{6i}(x)q_j\Bigg]\underline{b}_1$$

$$+ \Bigg[\phi_{5i}(x) + \sum_{j=1}^{\nu}\phi_{4j}(x)\phi_{6i}(x)q_j\Bigg]\underline{b}_2$$

$$+ \Bigg[\phi_{6i}(x) - \sum_{j=1}^{\nu}\phi_{4j}(x)\phi_{5i}(x)q_j\Bigg]\underline{b}_3,$$

$$(i = 1,\ldots,\nu) \quad (43)$$

while the linearized velocity of C in N and linearized angular velocity of dB in N can be expressed as

$$
N \underline{\tilde{v}}^C \overset{(32)}{=} \Bigg(v_1 + \omega_2 e_3 - \omega_3 e_2 + \sum_{k=1}^{\nu} \{ \omega_2 [\phi_{3k}(x) + e_2 \phi_{4k}(x)]
$$

$$
- \omega_3 [\phi_{2k}(x) - e_3 \phi_{4k}(x)] \} q_k
$$

$$
+ \sum_{k=1}^{\nu} \{ \phi_{1k}(x) - e_2 \phi_{6k}(x) + e_3 \phi_{5k}(x) \} \dot{q}_k \Bigg) \underline{a}_1
$$

$$
+ \Bigg(v_2 + \omega_3 x - \omega_1 e_3 + \sum_{k=1}^{\nu} \Big\{ \omega_3 [\phi_{1k}(x) - e_2 \phi_{6k}(x)
$$

$$
+ e_3 \phi_{5k}(x)] \Big) - \omega_1 [\phi_{3k}(x) + e_2 \phi_{4k}(x)] \Big\} q_k
$$

$$
+ \sum_{k=1}^{\nu} \{ \phi_{2k}(x) - e_3 \phi_{4k}(x) \} \dot{q}_k \Bigg) \underline{a}_2
$$

$$
+ \Bigg(v_3 + \omega_1 e_2 - \omega_2 x + \sum_{k=1}^{\nu} \Big\{ \omega_1 [\phi_{2k}(x) - e_3 \phi_{4k}(x)]
$$

$$
- \omega_2 [\phi_{1k}(x) - e_2 \phi_{6k}(x) + e_3 \phi_{5k}(x)] \Big\} q_k
$$

$$
+ \sum_{k=1}^{\nu} \{ \phi_{3k}(x) + e_2 \phi_{4k}(x) \} \dot{q}_k \Bigg) \underline{a}_3 \qquad (44)
$$

and

$$
N \underline{\tilde{\omega}}^{dB} \overset{(33)}{=} \Bigg\{ \omega_1 + \sum_{k=1}^{\nu} [\omega_2 \phi_{6k}(x) - \omega_3 \phi_{5k}(x)] q_k
$$

$$
+ \sum_{k=1}^{\nu} \phi_{4k}(x) \dot{q}_k \Bigg\} \underline{b}_1
$$

$$
+ \Bigg\{ \omega_2 + \sum_{k=1}^{\nu} [\omega_3 \phi_{4k}(x) - \omega_1 \phi_{6k}(x)] q_k + \sum_{k=1}^{\nu} \phi_{5k}(x) \dot{q}_k \Bigg\} \underline{b}_2
$$

$$
+ \Bigg\{ \omega_3 + \sum_{k=1}^{\nu} [\omega_1 \phi_{5k}(x) - \omega_2 \phi_{4k}(x)] q_k + \sum_{k=1}^{\nu} \phi_{6k}(x) \dot{q}_k \Bigg\} \underline{b}_3
$$

$$
(45)
$$

To produce linearized equations of motion, the expressions for $N\tilde{v}_i^C$, $N\tilde{\omega}_i^{dB}$, $N\tilde{v}^C$, and $N\tilde{\omega}^{dB}$ from Eqs. (42–45) will be used in conjunction with the relationship (Ref. 34, p. 158)

$$
\tilde{F}_i^* + \tilde{F}_i = 0, \qquad (i = 1, \dots, \nu) \qquad (46)
$$

where $\tilde{F}_i^*$ and $\tilde{F}_i$ are the ith linearized generalized inertia force and ith linearized generalized active force for the system, respectively. The linearized generalized inertia force $\tilde{F}_i^*$ can be written

$$
\tilde{F}_i^* = - \int_0^L \rho^N \underline{\tilde{v}}_i^C \cdot {}^N \underline{\tilde{a}}^C \, dx - \int_0^L {}^N \underline{\tilde{\omega}}_i^{dB} \cdot \Big({}^N \underline{\tilde{\alpha}}^{dB} \cdot \underline{\underline{I}}
$$

$$
+ {}^N \underline{\tilde{\omega}}^{dB} \times \underline{\underline{I}} \cdot {}^N \underline{\tilde{\omega}}^{dB} \Big) dx \qquad (i = 1, \dots, \nu) \quad (47)
$$

where $\rho \, dx$ is the mass of dB, $\underline{\underline{I}} \, dx$ represents the central inertia dyadic of dB, ${}^N \underline{\tilde{a}}^C$ denotes the linearized acceleration of C in N, ${}^N \underline{\tilde{\alpha}}^{dB}$ is the linearized angular acceleration of dB in N, and all other symbols have been defined previously. The central inertia dyadic can be represented as

$$
\underline{\underline{I}} \, dx \triangleq \rho (I_{11} \underline{b}_1 \underline{b}_1 + I_{22} \underline{b}_2 \underline{b}_2 + I_{33} \underline{b}_3 \underline{b}_3) \, dx \qquad (48)
$$

where I_{11}, I_{22}, I_{33}, expressed in terms of the central principal second moments of area I_2 and I_3, are given by

$$
I_{22} = \frac{I_2}{A_o}, \qquad I_{33} = \frac{I_3}{A_o}, \qquad I_{11} = \frac{I_2 + I_3}{A_o} = I_{22} + I_{33} \quad (49)
$$

Since only internal forces contribute to the generalized active force F_i, an expression for $\tilde{F}_i$ can be derived from the strain energy function U by taking advantage of the fact that

$$
F_i = - \frac{\partial U}{\partial q_i} \qquad (i = 1, \dots, \nu) \qquad (50)
$$

where U is given by

$$
U = \int_0^L \frac{(P_1)^2}{2 E_o A_o} \, dx + \int_0^L \frac{\alpha_2 (V_2)^2}{2 G A_o} \, dx + \int_0^L \frac{\alpha_3 (V_3)^2}{2 G A_o} \, dx
$$

$$
+ \int_0^L \frac{(T_1)^2}{2 G \kappa} \, dx + \int_0^L \frac{(M_2)^2}{2 E_o I_2} \, dx + \int_0^L \frac{(M_3)^2}{2 E_o I_3} \, dx \quad (51)
$$

and the symbols P_1, V_2, V_3, T_1, M_2, and M_3 are defined as follows. Replace the set S of all forces acting on the particular cross section of B which is located at a distance x from O when B is undeformed with a couple of torque $\underline{T}$ together with a force $\underline{R}$ whose line of action passes through the elastic center E of the cross section. Then $\underline{R}$ can be resolved into an axial component, $P_1 \underline{a}_1$, and a shear component, $V_2 \underline{a}_2 + V_3 \underline{a}_3$. Similarly, the torque $\underline{T}$ can be resolved into an axial torque, $T_1 \underline{a}_1$, and a bending moment, $M_2 \underline{a}_2 + M_3 \underline{a}_3$. All other symbols in Eq. (51), with the exception of κ have been defined previously. The symbol κ denotes an effective torsional constant[35,36] which would permit the same relative twist between beam ends for a given torque if one used the first-order relationship

$$
T_1 = G \kappa \frac{d\theta_1}{dx} \qquad (52)
$$

as would result from the use of the exact third-order equation for torsion,[37,38] namely,

$$
T_1 = G \kappa \frac{d\theta_1}{dx} - E_o \Gamma \frac{d^3 \theta_1}{dx^3} \qquad (53)
$$

where κ is the Saint Venant torsional factor for pure torsion and Γ is the warping factor introduced earlier. The effective torsional constant κ depends on the degree of fixity and warping at each end of the beam and can be calculated with the aid of Eqs. (52) and (53) in conjunction with tables in Ref. 32 (pp. 301–312), or by utilizing publicly available computer programs (SDRC FRAME,[39] SPOTS3[40]).

When effects of shear are included in the analysis, θ_2 and θ_3 are related to u_3 and u_2, respectively, as follows:

$$
\frac{\partial u_3(x, t)}{\partial x} = - [\theta_2(x, t) + \psi_2(x, t)] \qquad (54)
$$

and

$$
\frac{\partial u_2(x, t)}{\partial x} = [\theta_3(x, t) + \psi_3(x, t)] \qquad (55)
$$

where the symbols ψ_2 and ψ_3 denote angles of shear of a cross section, measured at the elastic axis and associated with $\underline{a}_2$ and $\underline{a}_3$, respectively. With these relationships, the shearing force measure numbers V_2 and V_3 appearing in Eq. (51) can be expressed as[41]

$$
V_2 = \frac{A_o G}{\alpha_2} \psi_3 = \frac{A_o G}{\alpha_2} \left[\frac{\partial u_2(x, t)}{\partial x} - \theta_3(x, t) \right] \quad (56)
$$

$$V_3 = -\frac{A_o G}{\alpha_3}\psi_2 = \frac{A_o G}{\alpha_3}\left[\frac{\partial u_3(x,t)}{\partial x} + \theta_2(x,t)\right] \quad (57)$$

The other force and moment measure numbers present in Eq. (51) are given by

$$P_1 = E_o A_o \frac{\partial s(x,t)}{\partial x}, \qquad T_1 = Gk\frac{\partial \theta_1(x,t)}{\partial x} \quad (58)$$

$$M_2 = E_o I_2 \frac{\partial \theta_2(x,t)}{\partial x}, \qquad M_3 = E_o I_3 \frac{\partial \theta_3(x,t)}{\partial x} \quad (59)$$

and substitution of these expressions for P_1, V_2, V_3, T_1, M_2, and M_3 into Eq. (51) yields

$$U = \frac{1}{2}\left\{\int_0^L E_o A_o\left[\frac{\partial s(x,t)}{\partial x}\right]^2 dx\right.$$

$$+ \int_0^L \frac{GA_o}{\alpha_2}\left[\frac{\partial u_2(x,t)}{\partial x} - \theta_3(x,t)\right]^2 dx$$

$$+ \int_0^L \frac{GA_o}{\alpha_3}\left[\frac{\partial u_3(x,t)}{\partial x} + \theta_2(x,t)\right]^2 dx$$

$$+ \int_0^L Gk\left[\frac{\partial \theta_1(x,t)}{\partial x}\right]^2 dx$$

$$+ \int_0^L E_o I_2\left[\frac{\partial \theta_2(x,t)}{\partial x}\right]^2 dx$$

$$\left. + \int_0^L E_o I_3\left[\frac{\partial \theta_3(x,t)}{\partial x}\right]^2 dx\right\} \quad (60)$$

If the expressions for s, u_2, u_3, θ_1, θ_2, and θ_3 from Eqs. (25-27) are now substituted into Eq. (60), and the resulting expression for U, in which the generalized coordinates $q_1,\ldots,q_\nu$ appear explicitly, is differentiated in the manner shown in Eq. (50), one finds

$$F_i = -\sum_{j=1}^{\nu}\left\{\int_0^L E_o A_o \phi'_{1i}(x)\phi'_{1j}(x)\,dx\right.$$

$$+ \int_0^L \frac{GA_o}{\alpha_3}\left[\phi_{5i}(x)\phi_{5j}(x) + \phi_{5i}(x)\phi'_{3j}(x)\right.$$

$$\left. + \phi_{5j}(x)\phi'_{3i}(x) + \phi'_{3i}(x)\phi'_{3j}(x)\right]dx$$

$$+ \int_0^L \frac{GA_o}{\alpha_2}\left[\phi_{6i}(x)\phi_{6j}(x) - \phi_{6i}(x)\phi'_{2j}(x)\right.$$

$$\left. + \phi_{6j}(x)\phi'_{2i}(x) + \phi'_{2i}(x)\phi'_{2j}(x)\right]dx$$

$$+ \int_0^L Gk\phi'_{4i}(x)\phi'_{4j}(x)\,dx$$

$$+ \int_0^L E_o I_2\phi'_{5i}(x)\phi'_{5j}(x)\,dx$$

$$\left. + \int_0^L E_o I_3\phi'_{6i}(x)\phi'_{6j}(x)\,dx\right\}q_j \quad (i=1,\ldots,\nu) \quad (61)$$

The term in braces in the above equation will be denoted by H_{ij}. Thus

$$\tilde{F}_i^{(61)} = -\sum_{j=1}^{\nu} H_{ij}q_j \quad (i=1,\ldots,\nu) \quad (62)$$

At this point in the derivation, equations of motion could be written in literal form by differentiating $^N\underline{v}^C$ and $^N\underline{\omega}^{dB}$ in N to obtain $^N\underline{a}^C$ and $^N\underline{\dot{\alpha}}^C$, substituting the resulting expressions,

together with the ones for $^N\underline{v}_i^C$, $^N\underline{\omega}_i^{dB}$, $^N\underline{\omega}^{dB}$, and $\underline{I}$, from Eqs. (42), (43), (45), and (48) into Eq. (47), and equating to zero the sum [see Eq. (46)] of the resulting generalized inertia force and the generalized active force given in Eq. (62). These operations lead to a set of linear dynamical equations that can be represented in matrix form as

$$M\ddot{q} + G\dot{q} + Kq = F \quad (63)$$

where M, G, and K are a mass matrix, skew-symmetric gyroscopic matrix, and stiffness matrix, respectively, all of dimension $\nu \times \nu$; $\ddot{q}$, $\dot{q}$, and q are $\nu \times 1$ column matrices having elements $\ddot{q}_i$, $\dot{q}_i$, and q_i, respectively, in the ith row; and F is a $\nu \times 1$ column matrix. (If viscous "modal damping" is desired, it can be accounted for in the usual manner by adding a term $C\dot{q}$ to the above equation, with C representing a symmetric matrix of suitable damping coefficients.) The four matrices M, G, K, and F are given explicitly in the next section.

IV. Simulation Algorithm

What follows is a description of a step-by-step procedure for simulating motions of the beam system described earlier, and thus determining values of $s(x,t)$, $u_2(x,t)$, $u_3(x,t)$, and $\theta_1(x,t)$ for $t > 0$, once all system parameters, base motion temporal functions, and the initial beam displacement and velocity spatial functions have been specified.

1) Specify beam length L, material properties $E_0(x)$, $G(x)$, $\rho(x)$ and cross-sectional properties $A_0(x)$, $I_2(x)$, $I_3(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $k(x)$, $e_2(x)$, and $e_3(x)$.

2) Specify base motion temporal functions $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$, $v_1(t)$, $v_2(t)$, and $v_3(t)$.

3) Specify initial displacement function $g(x)$ and initial velocity function $h(x)$ defined as

$$g(x) \triangleq \begin{bmatrix} s(x,0) \\ u_2(x,0) \\ u_3(x,0) \\ \theta_1(x,0) \\ \theta_2(x,0) \\ \theta_3(x,0) \end{bmatrix} \qquad h(x) \triangleq \begin{bmatrix} \dot{s}(x,0) \\ \dot{u}_2(x,0) \\ \dot{u}_3(x,0) \\ \dot{\theta}_1(x,0) \\ \dot{\theta}_2(x,0) \\ \dot{\theta}_3(x,0) \end{bmatrix} \quad (64)$$

4) Specify the number ν of assumed modal functions to be employed in the simulation.

5) Solve the eigenproblem associated with the cantilever beam with the root fixed in a Newtonian reference frame. Determine eigenfunctions $\phi_{ij}(x)$ ($i=1,\ldots,6$; $j=1,\ldots,\nu$) and eigenvalues λ_j ($j=1,\ldots,\nu$). (Note: For uniform beams of constant symmetrical cross section, one may wish to neglect shear effects and solve the associated eigenproblem analytically to obtain the necessary eigenfunctions and eigenvalues. For more complex beams, one should resort to a numerical technique, such as the finite element method, which yields discrete eigenvectors. These eigenvectors can be used in their discrete form in the subsequent calculations, or they can be converted into continuous functions through the use of shape functions.)

6) Calculate modal initial values as

$$q_i(0) = \sum_{j=1}^{6}\sum_{k=1}^{6}\int_0^L \phi_{ji}(x)\bar{E}_{jk}g_k\,dx, \quad (i=1,\ldots,\nu) \quad (65)$$

and

$$\dot{q}_i(0) = \sum_{j=1}^{6}\sum_{k=1}^{6}\int_0^L \phi_{ji}(x)\bar{E}_{jk}h_k\,dx, \quad (i=1,\ldots,\nu) \quad (66)$$

where $\bar{E}_{jk}$ is the element in the jth row and kth column of the inertia matrix $\bar{E}$ defined as

$$\bar{E} \triangleq \rho \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -e_3 & 0 & 0 \\ 0 & 0 & 1 & e_2 & 0 & 0 \\ 0 & -e_3 & e_2 & I_{11}+e_2^2+e_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{22}+e_3^2 & -e_2 e_3 \\ 0 & 0 & 0 & 0 & -e_2 e_3 & I_{33}+e_2^2 \end{bmatrix}$$

(67)

and g_k and h_k are the elements in the kth row of the respective column matrices g and h introduced in step 3.

7) Evaluate modal integrals:

$$\check{W}_{ki} \triangleq \int_0^L \rho \phi_{ki}(x)\,\mathrm{d}x$$

$$(i=1,\ldots,\nu;\ k=1,\ldots,6) \quad (68)$$

$$W_{klij} \triangleq \int_0^L \rho \phi_{ki}(x)\phi_{lj}(x)\,\mathrm{d}x$$

$$(i,j=1,\ldots,\nu;\ k,l=1,\ldots,6) \quad (69)$$

$$\check{X}_{ki} \triangleq \int_0^L \rho x \phi_{ki}(x)\,\mathrm{d}x \quad (i=1,\ldots,\nu;\ k=1,\ldots,6) \quad (70)$$

$$X_{klij} \triangleq \int_0^L \rho x \phi_{ki}(x)\phi_{lj}(x)\,\mathrm{d}x$$

$$(i,j=1,\ldots,\nu;\ k,l=1,\ldots,6) \quad (71)$$

$$\check{Y}_{ki} \triangleq \int_0^L \rho I_{22}\phi_{ki}(x)\,\mathrm{d}x \quad (i=1,\ldots,\nu;\ k=4,5,6) \quad (72)$$

$$Y_{klij} \triangleq \int_0^L \rho I_{22}\phi_{ki}(x)\phi_{lj}(x)\,\mathrm{d}x$$

$$(i,j=1,\ldots,\nu;\ k,l=4,5,6) \quad (73)$$

$$\check{Z}_{ki} \triangleq \int_0^L \rho I_{33}\phi_{ki}(x)\,\mathrm{d}x \quad (i=1,\ldots,\nu;\ k=4,5,6) \quad (74)$$

$$Z_{klij} \triangleq \int_0^L \rho I_{33}\phi_{ki}(x)\phi_{lj}(x)\,\mathrm{d}x$$

$$(i,j=1,\ldots,\nu;\ k,l=4,5,6) \quad (75)$$

$$\mu_{ij} \triangleq \int_0^L \rho(\beta_{ij}+\gamma_{ij})\,\mathrm{d}x \quad (i,j=1,\ldots,\nu) \quad (76)$$

$$\eta_{ij} \triangleq \int_0^L \rho x(\beta_{ij}+\gamma_{ij})\,\mathrm{d}x \quad (i,j=1,\ldots,\nu) \quad (77)$$

$$H_{ij} \triangleq \int_0^L \Bigg\{ E_o A_o \phi'_{1i}(x)\phi'_{1j}(x) + \frac{GA_o}{\alpha_2}\Big[\phi_{5i}(x)\phi_{5j}(x)$$

$$+\phi_{5i}(x)\phi'_{3j}(x)+\phi'_{3i}(x)\phi_{5j}(x)+\phi'_{3i}(x)\phi'_{3j}(x)\Big]$$

$$+\frac{GA_o}{\alpha_3}\Big[\phi_{6i}(x)\phi_{6j}(x)-\phi_{6i}(x)\phi'_{2j}(x)-\phi'_{2i}(x)\phi_{6j}(x)$$

$$+\phi'_{2i}(x)\phi'_{2j}(x)\Big]+GK\phi'_{4i}(x)\phi'_{4j}(x)$$

$$+E_o I_2\phi'_{5i}(x)\phi'_{5j}(x)+E_o I_3\phi'_{6i}(x)\phi'_{6j}(x)\Bigg\}\,\mathrm{d}x$$

$$(i,j=1,\ldots,\nu) \quad (78)$$

Table 1 Simplifications of general equations

To Neglect	Set [a]
Extension	$\phi_{1i}=0,\ i=1,\ldots,\nu$
Bending in a_2-dir.	$\phi_{2i}=0,\ i=1,\ldots,\nu$
Bending in a_3-dir.	$\phi_{3i}=0,\ i=1,\ldots,\nu$
Torsion	$\phi_{4i}=0,\ i=1,\ldots,\nu$
Rotatory inertia	$I_{22}=I_{33}=0$
Shear	$\alpha_2=\alpha_3=0$
Warping restraint	$\Gamma=0$
Nonuniform torsion	$\check{\kappa}=\kappa$

[a]Setting $\phi_{ni}=0\ (n=1,\ldots,6)$ implies setting $W_{klij}=X_{klij}=Y_{klij}=Z_{klij}=\check{W}_{ki}=\check{X}_{ki}=\check{Y}=\check{Z}_{ki}=0\ (k\ \text{or}\ l=n;\ i,j=1,\ldots,\nu)$, as well as setting all H_{ij} associated primarily with direction n equal to zero. [See Eqs. (68-75) and (78).]

(Note: Prior to evaluation of these integrals by numerical quadrature techniques, analytical integration by parts should be performed on the integrals appearing in Eqs. (76-77) in order to bring them into a simpler form. Additionally, the quantity H_{ij} in Eq. (78) usually can be evaluated by using modal orthogonality properties in such a way that integration becomes unnecessary. For instance, if the finite element method is employed in the eigensolution, and the resulting orthogonal eigenvectors are normalized with respect to the finite element mass matrix, then it can be shown that $H_{ij}=\lambda_j\delta_{ij}$, where δ_{ij} is the Kronecker delta.)

8) Assemble system mass, gyroscopic, stiffness, and force matrices:[**]

$$M_{ij} = \Bigg\{ \Big[\underline{W_{11ij}+W_{22ij}+W_{33ij}}$$

$$+Y_{44ij}+Y_{55ij}+Z_{44ij}+Z_{66ij}\Big]$$

$$+e_2\Big[(-W_{16ij}+W_{34ij}+W_{43ij}-W_{61ij})$$

$$+e_2(W_{44ij}+W_{66ij})\Big]$$

$$+e_3\Big[(W_{15ij}-W_{24ij}-W_{42ij}+W_{51ij})$$

$$+e_3(W_{44ij}+W_{55ij})\Big]-e_2 e_3\big[W_{56ij}+W_{65ij}\big]\Bigg\}$$

$$(i,j=1,\ldots,\nu) \quad (79)$$

$$G_{ij} = 2\Bigg\{ \omega_1\Big[\underline{W_{32ij}-W_{23ij}}\Big]$$

$$+\omega_2\Big[\underline{W_{13ij}-W_{31ij}}+Z_{46ij}-Z_{64ij}\Big]$$

$$+\omega_3\Big[\underline{W_{21ij}-W_{12ij}}+Y_{54ij}-Y_{45ij}\Big]$$

$$+e_2\big[\omega_1(W_{42ij}-W_{24ij})$$

$$+\omega_2(W_{14ij}+W_{36ij}-W_{41ij}-W_{63ij})$$

$$+\omega_3(W_{62ij}-W_{26ij})+e_2\omega_2(W_{46ij}-W_{64ij})\big]$$

$$+e_3\big[\omega_1(W_{43ij}-W_{34ij})+\omega_2(W_{53ij}-W_{35ij})$$

$$+\omega_3(W_{14ij}+W_{25ij}-W_{41ij}-W_{52ij})$$

$$+e_3\omega_3(W_{54ij}-W_{45ij})\big]-e_2 e_3\big[\omega_2(W_{45ij}-W_{54ij})$$

$$+\omega_3(W_{64ij}-W_{46ij})\big]\Bigg\} \quad (i,j=1,\ldots,\nu) \quad (80)$$

[**]The purpose of underlining certain terms is to facilitate the discussion in Sec. V concerning discrepancies between the present algorithm and various "multibody dynamics" algorithms.

$$K_{ij} = \Big\{ -\big(\omega_2^2 + \omega_3^2\big)W_{11ij} + \omega_1\omega_2\big(W_{12ij} + W_{21ij}\big)$$

$$+\omega_1\omega_3\big(W_{13ij} + W_{31ij}\big) - \big(\omega_1^2 + \omega_3^2\big)W_{22ij}$$

$$+\omega_2\omega_3\big(W_{23ij} + W_{32ij}\big) - \big(\omega_1^2 + \omega_2^2\big)W_{33ij}$$

$$-\big(\omega_2^2 - \omega_3^2\big)\big(Z_{44ij} - Y_{44ij}\big)$$

$$+\omega_2\omega_1\big(Z_{45ij} + Z_{54ij} - Y_{45ij} - Y_{54ij}\big)$$

$$+\omega_1\omega_3\big(Y_{46ij} + Y_{64ij} - Z_{46ij} - Z_{64ij}\big) - \big(\omega_3^2 - \omega_1^2\big)Y_{55ij}$$

$$+\omega_2\omega_3\big(Y_{56ij} + Y_{65ij}\big) - \big(\omega_2^2 - \omega_1^2\big)Z_{66ij}$$

$$-\dot\omega_1\Big[W_{23ij} - W_{32ij} + Y_{56ij} + Y_{65ij}\Big]$$

$$+\dot\omega_2\Big[W_{13ij} - W_{31ij} + Y_{46ij} + Y_{64ij} + Z_{46ij} - Z_{64ij}\Big]$$

$$-\dot\omega_3\Big[W_{12ij} - W_{21ij} + Y_{45ij} - Y_{54ij} + Z_{45ij} + Z_{54ij}\Big]$$

$$-\big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2\big)\mu_{ij} + \big(\omega_2^2 + \omega_3^2\big)\eta_{ij}$$

$$+e_2\Big[\omega_1\omega_3\big(W_{14ij} + W_{41ij}\big) + \big(\omega_2^2 + \omega_3^2\big)\big(W_{16ij} + W_{61ij}\big)$$

$$+\omega_2\omega_3\big(W_{24ij} + W_{42ij}\big) - \omega_1\omega_2\big(W_{26ij} + W_{62ij}\big)$$

$$-\big(\omega_1^2 + \omega_2^2\big)\big(W_{34ij} + W_{43ij}\big) - \omega_1\omega_3\big(W_{36ij} + W_{63ij}\big)$$

$$-\big(\dot v_2 + \omega_3 v_1 - \omega_1 v_3 - e_2\omega_3^2 + e_2\omega_2^2\big)W_{44ij}$$

$$+\big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2 + \omega_1\omega_2 e_2\big)\big(W_{45ij} + W_{54ij}\big)$$

$$-e_2\omega_1\omega_3\big(W_{46ij} + W_{64ij}\big)$$

$$-\big(\dot v_2 + \omega_3 v_1 - \omega_1 v_3 - e_2\omega_1^2 + e_2\omega_2^2\big)W_{66ij}$$

$$+\dot\omega_2\big(W_{14ij} - W_{41ij}\big) - \dot\omega_1\big(W_{24ij} - W_{42ij}\big)$$

$$-\dot\omega_3\big(W_{26ij} - W_{62ij}\big) + \dot\omega_2\big(W_{36ij} - W_{63ij}\big)$$

$$-\dot\omega_3 e_2\big(W_{45ij} + W_{54ij}\big) + \dot\omega_2 e_2\big(W_{46ij} - W_{64ij}\big)$$

$$+\big(\dot\omega_3 - \omega_1\omega_2\big)\mu_{ij} - \big(\dot\omega_3 + \omega_1\omega_2\big)X_{44ij}$$

$$-\big(\omega_2^2 + \omega_3^2\big)\big(X_{45ij} + X_{54ij}\big) - \big(\dot\omega_3 + \omega_1\omega_2\big)X_{66ij}\Big]$$

$$+e_3\Big[-\omega_1\omega_2\big(W_{14ij} + W_{41ij}\big)$$

$$-\big(\omega_2^2 + \omega_3^2\big)\big(W_{15ij} + W_{51ij}\big)$$

$$+\big(\omega_1^2 + \omega_3^2\big)\big(W_{24ij} + W_{42ij}\big)$$

$$+\omega_1\omega_2\big(W_{25ij} + W_{52ij}\big) - \omega_2\omega_3\big(W_{34ij} + W_{43ij}\big)$$

$$+\omega_1\omega_3\big(W_{35ij} + W_{53ij}\big)$$

$$-\big(\dot v_3 + \omega_1 v_2 - \omega_2 v_1 - e_3\omega_2^2 + e_3\omega_3^2\big)W_{44ij}$$

$$-e_3\omega_1\omega_2\big(W_{45ij} + W_{54ij}\big)$$

$$-\big(\dot v_3 + \omega_1 v_2 - \omega_2 v_1 - e_3\omega_1^2 + e_3\omega_3^2\big)W_{55ij}$$

$$+\big(\dot v_2 + \omega_3 v_1 - \omega_1 v_3 + \omega_2\omega_3 e_3\big)\big(W_{56ij} + W_{65ij}\big)$$

$$+\big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2 + \omega_1\omega_3 e_3\big)\big(W_{46ij} + W_{64ij}\big)$$

$$+\dot\omega_3\big(W_{14ij} - W_{41ij}\big) + \dot\omega_3\big(W_{25ij} - W_{52ij}\big)$$

$$-\dot\omega_1\big(W_{34ij} - W_{43ij}\big) - \dot\omega_2\big(W_{35ij} - W_{53ij}\big)$$

$$-e_3\dot\omega_3\big(W_{45ij} - W_{54ij}\big) + \dot\omega_2 e_3\big(W_{46ij} + W_{64ij}\big)$$

$$-\dot\omega_1 e_3\big(W_{56ij} + W_{65ij}\big) - \big(\dot\omega_2 + \omega_1\omega_3\big)\mu_{ij}$$

$$+\big(\dot\omega_2 - \omega_1\omega_3\big)\big(X_{44ij} + X_{55ij}\big) - \big(\omega_2^2 + \omega_3^2\big)\big(X_{46ij} + X_{64ij}\big)$$

$$+\big(\dot\omega_3 + \omega_1\omega_2\big)\big(X_{56ij} + X_{65ij}\big)\Big]$$

$$+e_2 e_3\Big[-4\omega_2\omega_3 W_{44ij} + 2\omega_1\omega_3\big(W_{45ij} + W_{54ij}\big)$$

$$+2\omega_1\omega_2\big(W_{46ij} + W_{64ij}\big) - \omega_2\omega_3 W_{55ij}$$

$$+\big(\omega_2^2 - \omega_1^2\big)\big(W_{56ij} + W_{65ij}\big) - \omega_2\omega_3 W_{66ij}$$

$$-\dot\omega_1\big(W_{55ij} - W_{66ij}\big) - \dot\omega_2\big(W_{45ij} - W_{54ij}\big)$$

$$+\dot\omega_3\big(W_{46ij} - W_{64ij}\big) + \dot\omega_2\big(W_{45ij} + W_{54ij}\big)$$

$$-\dot\omega_3\big(W_{46ij} + W_{64ij}\big)\Big] + H_{ij}\Big\} \qquad (i,j = 1,\dots,\nu) \quad (81)$$

$$F_i = -\Big\{ \big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2\big)\breve W_{1i} + \big(\dot v_2 + \omega_3 v_1 - \omega_1 v_3\big)\breve W_{2i}$$

$$+\big(\dot v_3 + \omega_1 v_2 - \omega_2 v_1\big)\breve W_{3i} + \omega_2\omega_3\big(\breve Z_{4i} - \breve Y_{4i}\big)$$

$$+\dot\omega_1\big(\breve Y_{4i} + \breve Z_{4i}\big) + \big(\dot\omega_2 + \omega_3\omega_1\big)\breve Y_{5i} + \big(\dot\omega_3 - \omega_1\omega_2\big)\breve Z_{6i}$$

$$-\big(\omega_2^2 + \omega_3^2\big)\breve X_{1i} + \big(\dot\omega_3 + \omega_1\omega_2\big)\breve X_{2i} - \big(\dot\omega_2 - \omega_1\omega_3\big)\breve X_{3i}$$

$$+e_2\Big[-\big(\dot\omega_3 - \omega_1\omega_2\big)\breve W_{1i} - \big(\omega_1^2 + \omega_3^2\big)\breve W_{2i}$$

$$+\big(\dot\omega_1 + \omega_2\omega_3\big)\breve W_{3i}$$

$$+\big(\dot v_3 + \omega_1 v_2 - \omega_2 v_1 + \omega_2\omega_3 e_2 + \dot\omega_1 e_2\big)\breve W_{4i}$$

$$-\big(\dot\omega_2 - \omega_1\omega_3\big)\breve X_{4i}$$

$$-\big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2 + \omega_1\omega_2 e_2 - \dot\omega_3 e_2\big)\breve W_{6i}$$

$$+\big(\omega_2^2 + \omega_3^2\big)\breve X_{6i}\Big]$$

$$+e_3\Big[\big(\dot\omega_2 + \omega_1\omega_3\big)\breve W_{i1} - \big(\dot\omega_1 - \omega_2\omega_3\big)\breve W_{2i} - \big(\omega_1^2 + \omega_2^2\big)\breve W_{3i}$$

$$-\big(\dot v_2 + \omega_3 v_1 - \omega_1 v_3 + \omega_2\omega_3 e_3 - \dot\omega_1 e_3\big)\breve W_{4i}$$

$$-\big(\dot\omega_3 + \omega_1\omega_2\big)\breve X_{4i}$$

$$+\big(\dot v_1 + \omega_2 v_3 - \omega_3 v_2 + \omega_1\omega_3 e_3 + \dot\omega_2 e_3\big)\breve W_{5i}$$

$$-\big(\omega_2^2 + \omega_3^2\big)\breve X_{5i}\Big]$$

$$+e_2 e_3\Big[\big(\omega_3^2 - \omega_2^2\big)\breve W_{4i} - \big(\dot\omega_3 - \omega_1\omega_2\big)\breve W_{5i}$$

$$-\big(\dot\omega_2 + \omega_1\omega_3\big)\breve W_{6i}\Big]\Big\}, \qquad (i = 1,\dots,\nu) \quad (82)$$

9) Solve the following system of equations for q.

$$M\ddot q + G\dot q + Kq = F \qquad (83)$$

10) Calculate physical displacements s, u_2, u_3, and θ_1 by using Eqs. (25–27).

All the equations in the above algorithm are written in a way that facilitates reduction of the general equations to deal with restricted cases. For instance, if the beam to be analyzed has a symmetric cross section, all terms in M, G, K, and F that are preceded by the scalar eccentricities e_2 and e_3 may be neglected. Additionally, if rotatory inertia is to be neglected, all modal integrals involving I_{22} and I_{33} will be zero, and system matrix terms including these integrals then need not be evaluated. Hence, by setting appropriate terms to zero in the complete equations provided here and subsequently

performing numerical simulations, one can recover the vast majority of previously published simulation results for beams. Table 1 shows how one proceeds to eliminate various beam effects. Note that, for purposes of clarity, the above equations and modal integrals are developed for a beam with constant eccentricity values. When the eccentricities are variable, they are simply included in the integrands of the modal integrals appearing in Eqs. (68–76).

V. Illustrative Example

The generality of the algorithm presented in Sec. IV permits one to obtain solutions to a wide range of problems that span diverse disciplines. One representative problem involving a beam subjected to general base motion is that of simulating the behavior of a space-based robotic manipulator, such as the one shown in Fig. 3, which consists of three links L_1, L_2, and L_3, connected by revolute joints.

The outboard link L_3 of the manipulator consists of a base A and two distinct segments B_1 and B_2. Segment B_1 is $2\frac{2}{3}$ meters long and has a symmetric box cross section, while B_2 is a $5\frac{1}{3}$-meter-long channel. Both segments are made of a material for which $E_0 = 6.895 \times 10^{10}$ N/m^2, $G = 2.6519 \times 10^{10}$ N/m^2, and $\rho/A_0 = 2766.67$ kg/m^3. The section properties for B_1 are $A_0 = 3.84 \times 10^{-4}$ m^2, $I_2 = 1.50 \times 10^{-7}$ m^4, $I_3 = 1.50 \times 10^{-7}$ m^4, $\alpha_2 = 2.09$, $\alpha_3 = 2.09$, $\kappa = 2.2 \times 10^{-7}$ m^4, $\Gamma = 0$, $e_2 = e_3 = 0$, while the corresponding properties for B_2 are $A_0 = 7.3 \times 10^{-5}$ m^2, $I_2 = 4.8746 \times 10^{-9}$ m^4, $I_3 = 8.2181 \times 10^{-9}$ m^4, $\alpha_2 = 3.174$, $\alpha_3 = 1.52$, $\kappa = 2.4330 \times 10^{-11}$ m^4, $\Gamma = 5.0156 \times 10^{-13}$ m^6, $e_2 = 0$, and $e_3 = 0.01875$ m. To demonstrate the use of the simulation algorithm, we examine the behavior of L_3 during deployment of the manipulator from a stowed configuration to a fully operational configuration. The deployment process is presumed to last for 15 seconds ($T = 15$), during which time the values of the angles Ψ_1, Ψ_2, and Ψ_3, shown in Fig. 3, change from 180 to 90 deg, 180 to 45 deg, and 180 to 90 deg, respectively. If the two inboard links L_1 and L_2, each of length ℓ, are treated as rigid, this maneuver results in a prescribed motion of the base A of the outboard link, a motion characterized by the following six temporal functions:

$$v_1 = \ell(1 + \bar{c}_2)\bar{s}_3\dot{\Psi}_1, \qquad v_2 = \ell\dot{\Psi}_2, \qquad v_3 = -\ell(1 + \bar{c}_2)\bar{c}_3\dot{\Psi}_1$$
(84)

$$\omega_1 = \dot{\Psi}_1\bar{s}_2\bar{c}_3 - \dot{\Psi}_2\bar{s}_3, \qquad \omega_2 = \dot{\Psi}_1\bar{c}_2 + \dot{\Psi}_3,$$

$$\omega_3 = \dot{\Psi}_1\bar{s}_2\bar{s}_3 + \dot{\Psi}_2\bar{c}_3$$
(85)

where

$$\bar{c}_i \triangleq \cos\Psi_i, \qquad \bar{s}_i \triangleq \sin\Psi_i \qquad (i = 1,2,3)$$
(86)

We take $\ell = 8$ m, and let

$$\Psi_1(t) = \begin{cases} \pi - \dfrac{\pi}{2T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right) \text{ rad} & 0 < t \le T \\ \dfrac{\pi}{2} \text{ rad} & t > T \end{cases}$$
(87)

$$\Psi_2(t) = \begin{cases} \pi - \dfrac{3\pi}{4T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right) \text{ rad} & 0 < t \le T \\ \dfrac{\pi}{4} \text{ rad} & t > T \end{cases}$$
(88)

$$\Psi_3(t) = \begin{cases} \pi - \dfrac{\pi}{T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right) \text{ rad} & 0 < t \le T \\ 0 \text{ rad} & t > T \end{cases}$$
(89)

Figure 4 shows, for the tip of B_2, the transverse displacements u_2 and u_3, and the twist θ_1 during a time interval of 30 s, which corresponds to twice the deployment time. (The first

10 modes of the eigensolution were used to obtain these results.)

VI. Conventional Approach

The purpose of this section is to bring to light certain deficiencies of various existing multibody dynamic simulation computer programs, such as DISCOS,[26] NBOD2,[27] ALLFLEX,[29] and TREETOPS,[30] and to assess the significance of these deficiencies in connection with practical problems. Toward these ends, we first examine a fundamental issue
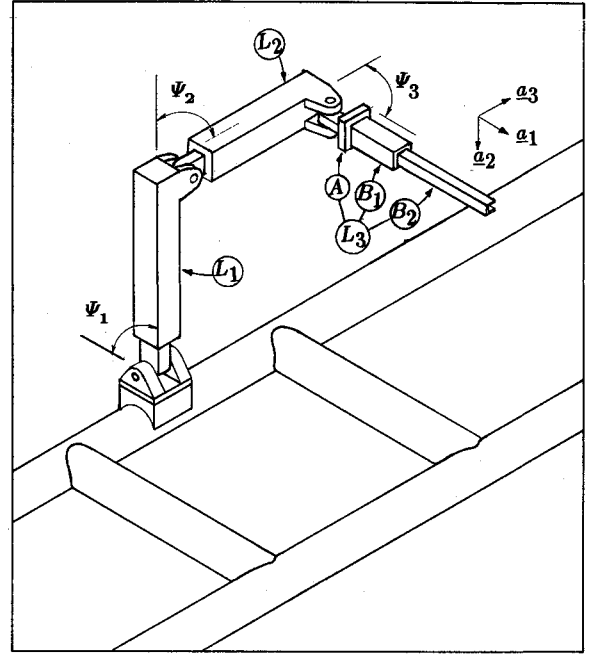

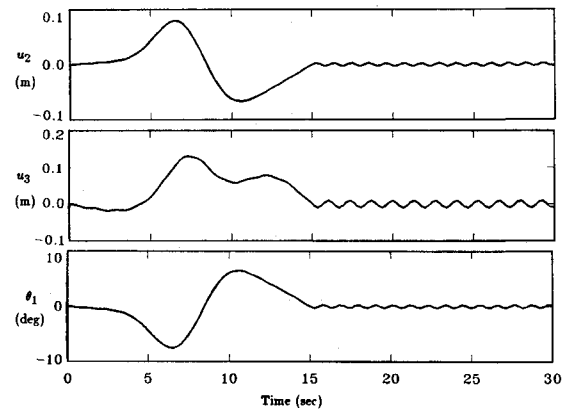Fig. 3   Space-based robotic manipulator.

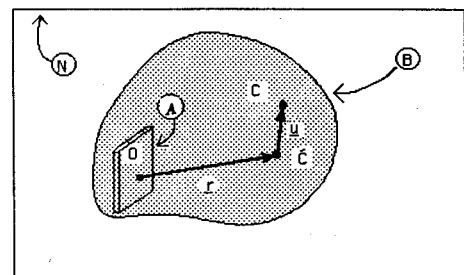Fig. 4   Manipulator deployment simulation results.

Fig. 5   Generic body of multibody system.

that arises when one seeks to formulate equations to be used for the simulation of large overall motions of systems containing deformable bodies that experience only "small" deformations. Next, we comment briefly on matters of concern solely in connection with beams. Finally, we present and discuss numerical results for both a repositioning maneuver and a spin-up maneuver of a flexible beam attached to a rigid base. The system consisting of a flexible beam attached to a rigid base constitutes one of the simplest systems within the application range of multibody programs and thus serves here as a "model" problem with which to check multibody dynamics formalisms.

While the derivations of the equations of motion underlying the aforementioned multibody programs differ from each other in many details, they all rely on a common approach to characterize "small" deformations of individual flexible bodies. This *conventional* approach, as we shall call it, involves developing an expression for the velocity of a generic particle, say C, of a deformable body B (Fig. 5) by writing

$$^N\underline{v}^C = {}^N\underline{v}^O + {}^N\underline{\omega}^A \times (\underline{r} + \underline{u}) + {}^A\underline{v}^C \qquad (90)$$

where N is a Newtonian reference frame; O is the point at which B is connected to another body of the system; A is a reference frame in which a differential element of B containing O is fixed; $^N\underline{v}^O$ is the velocity of O in N, $^N\underline{\omega}^A$ is the angular velocity of A in N, $\underline{r}$ is the position vector from O to C, that point of A with which C coincides when B is undeformed; $\underline{u}$ is the displacement vector extending from C to C; and $^A\underline{v}^C$ is the velocity of C in A. Once a set of unit vectors $\underline{a}_1, \underline{a}_2, \underline{a}_3$ fixed in A has been introduced, the vectors $\underline{r}$ and $\underline{u}$ are written

$$\underline{r} = x_1\underline{a}_1 + x_2\underline{a}_2 + x_3\underline{a}_3 \qquad (91)$$

$$\underline{u} = u_1\underline{a}_1 + u_2\underline{a}_2 + u_3\underline{a}_3 \qquad (92)$$

and u_1, u_2, and u_3 are expressed as [see Eq. (II-24) in Ref. 26, Eq. (43) in Ref. 27, Eqs. (6) and (8) in Ref. 29, and Eq. (2) in Ref. 30]

$$u_i = \sum_{j=1}^{\nu} \phi_{ij}(x_1, x_2, x_3) q_j(t) \qquad (i = 1, 2, 3) \qquad (93)$$

where ν is the number of terms in the series expansion, $\phi_{ij}(x_1, x_2, x_3)$ $(i = 1, 2, 3; \; j = 1, \dots, \nu)$ are *independent* spatial (modal) functions, and $q_j(t)$ $(j = 1, \dots, \nu)$ are generalized coordinates. It follows immediately that

$$\dot{u}_i = \sum_{j=1}^{\nu} \phi_{ij}(x_1, x_2, x_3) \dot{q}_j(t) \qquad (i = 1, 2, 3) \qquad (94)$$

so that, for $i = 1$, one has

$$\dot{u}_1 = \sum_{j=1}^{\nu} \phi_{1j}(x_1, x_2, x_3) \dot{q}_j(t) \qquad (95)$$

Now, this relationship, which is intended to be valid for a "general" structure, is in direct conflict with Eq. (29), developed specifically for a beam. The source of the conflict is the absence from the conventional approach of a counterpart to Eq. (19), which reflects an important physical fact, namely, that every transverse displacement of a point on the neutral axis of a beam (that is, a nonzero value of either u_2 or u_3) gives rise to an axial displacement (that is, a nonzero value of u_1). In other words, the conventional approach treats u_1 exactly like u_2 and u_3, which is precisely what is wrong with it. The failure of the conventional approach to deal adequately with the interdependence of displacements ultimately leads to a theory in which the counterpart to Eq. (81) lacks the terms

involving μ_{ij} and η_{ij}. We shall discuss the practical significance of this difference between the present theory and that corresponding to the conventional approach presently. First, however, we wish to point out additional differences between the two theories.

In the formalisms underlying the aforementioned multibody programs [see Eqs. (II-18a) and (II-21) in Ref. 26, Eqs. (20) and (21) in Ref. 27, Eq. (22) in Ref. 29, and Eq. (7) in Ref. 30], as well as in the derivation underlying the present theory [see Eq. (47)], the development of expressions for inertia forces for a flexible body necessitates either an implied or an explicit integration over the space occupied by the body. When this space is three-dimensional, it is appropriate for the integrand to involve solely quantities associated with translation of a generic point of the body. But when the space is one-dimensional or two-dimensional, as in connection with beams, plates, shells, etc., provisions must be made not only for translation of a generic point of a differential element, but also for *rotation* of the differential element. It is precisely for this reason that we introduce the angular velocity $^N\underline{\omega}^{dB}$ in Eq. (17). The equation derivations in Refs. 26, 29, and 30 fail to incorporate such a term. If this term is absent also from a computer program based on one of these references, then rotational inertia effects associated with deformable structural elements are neglected in this program. For beams, this means that torsional effects are neglected entirely, which is a serious matter when one is dealing with beams connected to each other. Also, it precludes the accommodation of rotatory inertia effects, but this is relatively unimportant since often these effects are quite small.

Additional differences between the present theory and others reflect refinements rather than conflicts. Specifically, provisions are made in the present theory to deal with shear and warping by employing the concepts of a shear area ratio and an effective torsional factor calculated with the aid of a warping factor; coupling of bending and torsion is made possible through the introduction of an eccentricity vector extending from the shear center to the centroid of a cross
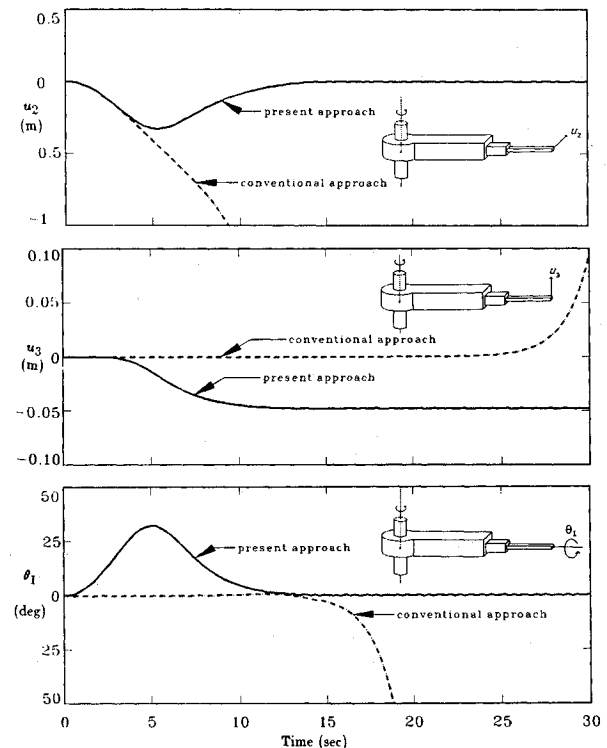

Fig. 6   Spin-up maneuver.

section. These effects are not considered in the conventional approach.

In summary, to see the difference in final equation form between the present theory and that corresponding to the conventional approach applied to a beam, one can refer to Eqs. (79–82), where the doubly underlined terms represent the conventional theory. The remaining terms in these equations account for dynamic stiffness effects (μ_{ij} and η_{ij}), torsion (W_{44ij}, X_{44ij}, Y_{44ij}, Z_{44ij}, and $\check{W}_{4i}$, $\check{X}_{4i}$, $\check{Y}_{4i}$, $\check{Z}_{4i}$), rotatory inertia (Y_{55ij} and Z_{66ij}), coupling of bending and torsion (e_2 and e_3 terms, as well as terms with indices 4,5 or 4,6), and coupling of differential element rotations θ_2 and θ_3 [all terms in Eqs. (81) and (82) with indices $k, l = 5,6$]. While many of the existing multibody programs may, in fact, incorporate the torsional terms mentioned above, *none* of these programs accommodate the dynamic stiffness effects that are crucial for certain simulations.

While the comparison of equations is enlightening, it furnishes no indication regarding the quantitative significance of the terms one loses when taking the conventional approach. Therefore, we consider a numerical example which, it turns out, permits one to reach some conclusions regarding this matter.

Returning to the manipulator system introduced earlier, we now examine the response of the tip of B_2 when Ψ_1, Ψ_2, and Ψ_3 (see Fig. 3) are given by

$$\Psi_1(t) = \frac{\pi}{2}\,\text{rad}, \qquad t > 0 \tag{96}$$

$$\Psi_2(t) = \begin{cases} \dfrac{6}{15}\left[\dfrac{t^2}{2} + \left(\dfrac{15}{2\pi}\right)^2\left(\cos\dfrac{2\pi t}{15} - 1\right)\right]\text{rad}, \\ \hspace{3cm} 0 < t \le 15\ \text{s} \\ (6t - 45)\ \text{rad}, \qquad t > 15\ \text{s} \end{cases} \tag{97}$$

$$\Psi_3(t) = 0\ \text{rad} \qquad t > 0 \tag{98}$$

During the corresponding manipulator motion, which we call a spin-up maneuver, link L_1 remains at rest, L_2 and A remain at rest relative to each other, and the angular speed of A, namely, $\dot{\Psi}_2$, increases from zero to 6 rad/s in a time interval of 15 s, and remains constant thereafter. (The manner in which L_1 and L_2 are shown to be attached to each other precludes this sort of motion, but this is irrelevant for present purposes.) Such a motion is of practical interest in connection with the operation of helicopter rotor blades, aircraft propellors, satellites with flexible appendages, etc. The behavior of link L_3 during this maneuver was predicted using two Fortran computer simulation programs. One program is based on the present theory and includes the full equations given earlier as Eqs. (79–83). The second program is based on the conventional approach discussed above; in other words, it includes only those terms in Eqs. (79–82) which are doubly underlined. For the particular spin-up maneuver mentioned, the conventional approach leads to the dashed curves for u_2, u_3, and θ_1 in Fig. 6, which indicate a cataclysmic instability of the motion under consideration, whereas use of our theory produces the solid curves in the same figure, which show just the opposite, namely, an entirely well-behaved response. On physical grounds, the latter is precisely what is to be expected, for we are dealing here with a simple case of "centrifugal stiffening." Thus one is forced to conclude that the conventional approach can lead to totally misleading predictions of the behavior of elastic bodies in certain situations.

VII. Conclusion

A comprehensive theory for dealing with small vibrations of a general beam attached to a base that is undergoing an arbitrary, prescribed motion has been presented. Numerical results show that predictions of this theory can differ markedly from those obtained by using an algorithm based on the conventional approach underlying various multibody dynamics programs. The present theory has been extended to incorporate "free" base motion as well; this work will be discussed in a forthcoming paper dealing with the attitude behavior of an elastic satellite subjected to gravitational and control forces. In an effort to develop a capability to simulate multibody systems correctly with beam-like components, the theory presented here is currently being incorporated into a new multibody computer program. We believe that recent advances in the use of simple continuum models in conjunction with modal parameter identification methods warrant such an extensive and comprehensive development activity. Certain features of the present derivation, such as the inclusion of differential element rotation, facilitate the process of connecting flexible bodies, and thus render this theory ideally suited for inclusion in such programs.

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